WIENER TRANSFORMATION ON FUNCTIONS
WITH BOUNDED AVERAGES

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ABSTRACT. The Wiener transformation (integrated Fourier transformation) on
the space $B^2 = \{f\colon \|f\| = \sup_{T \geq 1} (\frac{1}{2T} \int_{-T}^{T} |f|^2)^{1/2} < \infty\}$ is studied.

1. INTRODUCTION

In the celebrated paper on generalized harmonic analysis [12], Wiener proved
the following identity

$$(1.1) \qquad \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(x)|^2 \, dx = \lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \int_{-\infty}^{\infty} |\Delta_\varepsilon g(u)|^2 \, du,$$

where $g = W(f)$ is the Wiener transformation (integrated Fourier transforma-
tion) of f defined by

$$(1.2) \qquad g(u) = \frac{1}{2\pi} \left(\int_{-\infty}^{-1} + \int_{1}^{\infty} f(x) \frac{e^{iux}}{ix} \, dx + \int_{-1}^{1} f(x) \frac{e^{iux}}{ix} \, dx \right)$$

and $\Delta_\varepsilon g(u) = g(u + \varepsilon) = g(u - \varepsilon)$. He then used the identity to study the
almost periodic functions, and the spectrum and ergodicity of sample paths in
his pioneer work of stochastic processes.

The class of functions in (1.1), however, is not closed under addition. Two
natural Banach spaces to be considered in this respect are

$$B^p = \left\{ f \in L_{loc}^p(R)\colon \|f\|_{B^p} = \sup_{T \geq 1} \left(\frac{1}{2T} \int_{-T}^{T} |f(x)|^p \, dx \right)^{1/p} < \infty \right\},$$

and

$$V^P \left\{ g \in L_{loc}^p(R)\colon \|g\|_{V^p} = \sup_{1 \geq \varepsilon > 0} \left(\frac{1}{2\varepsilon} \int_{-\infty}^{\infty} |\Delta_\varepsilon g(u)|^p \, du \right)^{1/p} < \infty \right\},$$

where $1 < p < \infty$. In this case, functions in (1.1) can be considered as sub-classes of the quotient spaces B^2/B_0^2 and V^2/V_0^2, where

$$B_0^p = \left\{ f \in B^p : \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^T |f(x)|^p = 0 \right\},$$

and

$$V_0^p = \left\{ g \in V^p : \lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \int_{-\infty}^{\infty} |\Delta_\varepsilon g(u)|^p \, du = 0 \right\}.$$

The spaces B^p had been considered by Beurling in [3]. He defined a convolution algebra A^p, $1 \le p < \infty$, by

$$A^p = \left\{ f : \|f\|_{A^p} = \int_{\omega \in \Omega} \left(\int |f|^p \omega^{-(p-1)} \right)^{1/p} < \infty \right\},$$

where Ω is the set of bounded, positive, integrable even functions ω which are nonincreasing on R^+ and

$$\omega(0) + \int_{-\infty}^{\infty} \omega(x) \, dx = 1.$$

It is easy to show that $A^1 = L^1$, and A^p can be continuously embedded into L^1 and L^p.

Theorem 1.1 (Beurling). *For $1 \le p < \infty$, $1/p + 1/q = 1$, A^p is a convolution algebra, and $(A^p)^*$ is isomorphic to B^q.*

By regarding A^p, B^q as an L^1, L^∞ analog (rather than the L^p, L^q analog), Chen and Lau [4] defined a class of functions,

$$\mathrm{CMO}^p = \left\{ f : \|f\|_{*,p} = \sup_{T \ge 1} \left(\frac{1}{2T} \int_{-T}^T |f - m_T f|^p \right)^{1/p} < \infty \right\},$$

where $m_T f = (1/2T) \int_{-T}^T f$, $1 < p < \infty$. (CMO stands for *Central Mean Oscillation*.) The average in the norm is the p-th variance of the function f. It is analogous to BMO, but takes average only on intervals $[-T, T]$. If we let H_{A^p} be the corresponding Hardy space of A^p, then

Theorem 1.2. *For $1 < p < \infty$, $1/p + 1/q = 1$, $(H_{A^p})^* = \mathrm{CMO}^q$.*

Theorem 1.3. *For $1 < p < \infty$, and for any real f, $f + i\tilde{f} \in H_{A^p}$ if and only if $f^* \in A^p$, where $\tilde{f}$ is the conjugate of f, and f^* is the nontangential maximal function of f.*

The case $1 < p \le 2$ was proved in [4], and the general case was obtained by Garcia-Cuerva in [6]. Note that for $p = 1$, Theorem 1.2 is the Fefferman–Stein's duality theorem, and Theorem 1.3 is the Burkholder, Gundy, Silverstein's maximal function characterization of H^1.

While the space B^p enjoys the properties of L^∞ under duality and Hilbert transformation, we will show in this paper that B^2, on the other hand, will behave like L^2 in connection with the Fourier transformation and the Plancherel Theorem.

In the Schwartz distributional sense, the Wiener transformation of f satisfies $(Wf)' = \check{f}$ where $'$ is the derivative, and $\check{}$ denotes the inverse Fourier transformation (Proposition 3.1). We prove

Theorem 1.4. *The Wiener transformation $W: B^2 \to V^2$ is an isomorphism with*

$$\|W\| = \left(\tilde{h}(0) + \int_1^\infty \tilde{h}(x)\,dx\right)^{1/2}, \|W^{-1}\| = (h(1))^{-1/2},$$

where

$$h(x) = \frac{2}{\pi}\left(\frac{\sin x}{x}\right)^2; \quad x > 0 \text{ and } \tilde{h}(x) = \sup_{t \geq x} h(t).$$

The isomorphism can be rewritten as

$$c_2 \sup_{T \geq 1} \frac{1}{2T}\int_{-T}^T |f|^2 \leq \sup_{0 < \varepsilon \leq 1}\int_{-\infty}^\infty |\Delta_\varepsilon g|^2 \leq c_1 \sup_{T \geq 1}\frac{1}{2T}\int_{-T}^T |f|^2, \quad \forall f \in B^2.$$

It is an extension of (1.1), and is also an extension of Theorem 5.2 in [9] where the limit supremum is considered. The proof follows from some more general inequalities which will have independent interest in connection with ergodic theory.

2. SOME INEQUALITIES

For a nonnegative continuous function h on $[0, \infty)$, we will let $\tilde{h}(x) = \sup_{t \geq x} h(t)$, the smallest decreasing majorant of h.

Theorem 2.1. *Let h be a nonnegative continuous function on $[0, \infty)$ such that $c_1 = \tilde{h}(0) + \int_1^\infty \tilde{h}(x)\,dx < \infty$. Then for any nonnegative Borel measurable f,*

$$\sup_{T \geq 1}\int_0^\infty f(Tx)h(x)\,dx \leq c_1 \sup_{T \geq 1}\frac{1}{T}\int_0^T f(x)\,dx,$$

and c_1 is the best estimate for the inequality.

Proof. The inequality is obvious if the right side is infinite, hence we may assume without loss of generality that $\sup_{T \geq 1}(1/T)\int_0^T f(x)\,dx = 1$. It follows that

(2.1) $$\int_0^x f(Tt)\,dt \leq x, \quad T \geq 1, \quad x \geq 1.$$

For any $b > 1$, $T \geq 1$,

$$\int_0^b f(Tx)\tilde{h}(x)\,dx$$

$$= \int_0^1 f(Tx)\tilde{h}(x)\,dx + \left(\tilde{h}(b)\int_0^b f(Tx)\,dx - \tilde{h}(1)\int_0^1 F(T_x)\,dx\right.$$

$$\left. - \int_1^b \left(\int_0^x f(Tt)\,dt\right)d\tilde{h}(x)\right)$$

by integration by parts on $\int_1^b$

$$\leq (\tilde{h}(0) - \tilde{h}(1))\int_0^1 f(Tx)\,dx + \left(\tilde{h}(b)b - \int_1^b x\,d\tilde{h}(x)\right)$$

by (2.1) and the decreasing property of $\tilde{h}$

$$\leq (\tilde{h}(0) - \tilde{h}(1)) + \left(\tilde{h}(1) + \int_1^b \tilde{h}(x)\,dx\right)$$

$$= \tilde{h}(0) + \int_1^b \tilde{h}(x)\,dx.$$

By letting $b \to \infty$, the inequality follows. The following lemma will imply that c_1 is the best estimate.

Lemma 2.2. *Let h be as above, then for any $\varepsilon > 0$, there exists f such that* $\sup_{T \geq 1}(1/T)\int_0^T f \leq 1$, *and*

$$\sup_{T \geq 1}\int_0^\infty f(Tx)h(x)\,dx \geq \tilde{h}(0) + \int_1^\infty \tilde{h}(x)\,dx - \varepsilon.$$

Proof. Let x_0 be the largest x such that $h(x) = \tilde{h}(0)$. Let $s = \max\{1, x_0\}$, and let $A = \{x > s: h(x) \neq \tilde{h}(x)\}$. Then A is the disjoint union of a sequence of intervals $\{(a_i, b_i)\}$ such that $\max_{a_i \leq x \leq b_i} h(x) = h(b_i)$. For simplicity, we let $A = (a, b)$, the general case is similar.

Let

$$\eta = \int_s^\infty (\tilde{h}(x) = h(x)\,dx = \left(h(b)(b - a) - \int_a^b h(x)\,dx\right).$$

For $0 < \varepsilon < \eta$, by the continuity of h and the choice of b, we can find ε_1, $\delta > 0$ such that

$$\eta - \varepsilon < (h(b) - \varepsilon_1)(b - a) - \int_a^{b+\delta} h(x)\,dx$$

$$0 < h(b) - \varepsilon_1 \leq h(x), \qquad \forall b \leq x \leq b + \delta,$$

$$0 < h(x_0) - \varepsilon \leq h(x), \qquad \forall x_0 \leq x \leq x_0 + \delta,$$

$$0 < \int_{x_0}^{x_0+\delta} h(x)\,dx < \varepsilon,$$

and that $[x_0, x_0 + \delta] \cap [b, b + \delta] = \varnothing$. Define

$$
f_1(x) = \begin{cases} \frac{\max\{1, x_0\}}{\delta}, & x \in [x_0, x_0 + \delta), \\ \frac{b-a}{\delta}, & x \in [b, b + \delta), \\ 1, & x \in [s, \infty) \setminus ([x_0, x_0 + \delta) \cup [a, b + \delta)), \\ 0, & \text{otherwise.} \end{cases}
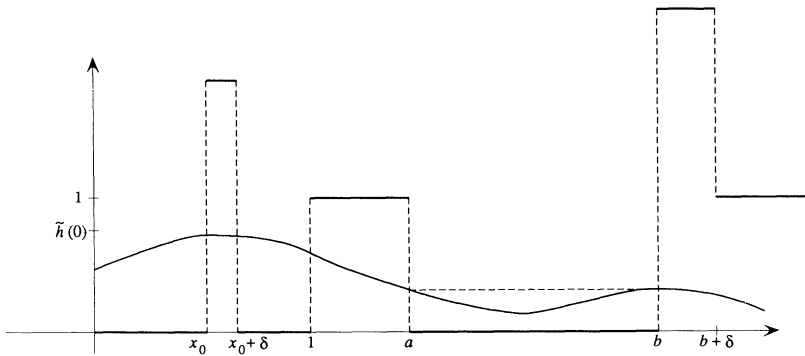$$

FIGURE 1.

(See Figure 1 for the case $x_0 < 2$.) It is a direct calculation that $\sup_{T \geq 1}(1/T) \int_0^T f_1 \leq 1$, and

$$
\int_0^\infty f_1(x) h(x) \, dx \geq \tilde{h}(0) + \int_0^\infty \tilde{h}(x) \, dx = k\varepsilon.
$$

where k can be predetermined ($k = 2 + x_0$ for the case $x_0 \geq 1$, and $k = 2$ for the case $x_0 < 1$).

For the reverse inequality, we have the following:

Theorem 2.3. *Let h be a nonnegative continuous function on $[0, \infty)$ such that for $x \in [0, 1]$, $h(x) \geq h(1) = c_2$, and $xh(x) \leq c_2$, then for any nonnegative Borel measurable f on $[0, \infty)$,*

$$
c_2 \sup_{T \geq 1} \frac{1}{T} \int_0^T f(x) \, dx \leq \sup_{T \geq 1} \int_0^\infty f(Tx) h(x) \, dx.
$$

Moreover, c_2 is the best estimate for the inequality.

Proof. Note that for $T \geq 1$,

$$c_2 \frac{1}{T} \int_0^T f(x)\,dx = \int_0^1 f(Tx)h(1)\,dx$$

$$\leq \int_0^1 f(Tx)h(x)\,dx$$

$$\leq \int_0^\infty f(Tx)h(x)\,dx.$$

The inequality follows by taking supremum for $T \geq 1$ on both sides. To show that c_2 is the best estimate, we will construct for any given $\varepsilon > 0$, an f_2 such that $\sup_{T \geq 1}(1/T)\int_0^T f_2 = 1$ and

$$\sup_{T \geq 1} \int_0^\infty f_2(Tx)h(x)\,dx \leq c_2 + \varepsilon.$$

Let $0 < \delta < \frac{1}{2}$ be such that $(1/(1-\delta))c_2 \leq c_2 + \dot{\varepsilon}$, and let

$$f_2 = \frac{1}{\delta}\chi_{[1-\delta,1]}.$$

A direct calculation will show that f_2 is the required function.

Remark 2.1. The continuity of h in the above theorems is only used to obtain the best constants for the estimates.

Remark 2.2. The function $h(x) = |\sin x/x|^p$, $p > 1$, and the Poisson kernel $p(x) = 1/\pi(1 + x^2)$ satisfies Theorem 2.1, and Theorem 2.3. The inequalities for $h(x) = |\sin x/x|^2$ will be used in the next section. The case for $p(x)$ can be used to estimate the harmonic extensions of f [4].

Remark 2.3. By assuming that h satisfies conditions in Theorem 2.1 and that there exists x_0 such that

$$(2.2) \quad \max_{x>0} xh(x) = x_0 h(x_0) = c_2' \quad \text{and} \quad h(x) \geq h(x_0), \quad \forall x \in [0, x_0],$$

the above proof can be adjusted to show that

$$c_2' \varlimsup_{T \to \infty} \frac{1}{T} \int_0^T f(x)\,dx \leq \varlimsup_{T \to \infty} \int_0^\infty f(Tx)h(x)\,dx \leq c_1' \varlimsup_{T \to \infty} \frac{1}{T} \int_0^T f(x)\,dx$$

for any nonnegative Borel measurable f on $[0, \infty)$, and that $c_1' = \int_0^\infty \tilde{h}(x)\,dx$, c_2' are sharp constants [9, Theorem 4.5, Theorem 4.6].

Remark 2.4. Let h be a complex valued function on $[0, \infty)$ such that
   (i) h has finite variation on any closed subintervals of $[0, \infty)$,
   (ii) $\int_0^\infty |h|^\sim(x)\,dx < \infty$.

It was proved in [8, Lemma 3.2] that

$$\lim_{T \to \infty} \int_0^\infty f(Tx)h(x)\,dx = c \lim_{T \to \infty} \frac{1}{T} \int_0^T f(x)\,dx\,,$$

where $c = \int_0^\infty h(x)\,dx$, provided that $\lim_{T \to \infty}(1/T)\int_0^T f(x)\,dx$ exists.

3. WIENER TRANSFORMATION ON B^2

We will use the following convention: the Fourier transformation of f is defined by

$$\hat{f}(u) = \int_{-\infty}^\infty f(x)e^{-iux}\,dx\,,$$

and the inverse Fourier transformation of g is

$$\check{g}(x) = \frac{1}{2\pi} \int_{-\infty}^\infty g(u)e^{iux}\,du\,.$$

Also we let

$$\Delta_\varepsilon g(x) = g(x+\varepsilon) - g(x-\varepsilon)\,,$$
$$\Delta_\varepsilon^+ g(x) = g(x+\varepsilon) - g(x)\,,$$
$$\Delta_\varepsilon^- g(x) = g(x) - g(x-\varepsilon)\,.$$

The space B^2 is contained in $L^2(R,\ dx/(1+x^2))$ [9, Proposition 2.1], hence for $f \in B^2$, the integral

$$\int_{-\infty}^{-1} + \int_1^\infty \frac{|f(x)|^2}{x^2}\,dx$$

exists. This implies that

$$\int_{-\infty}^{-1} + \int_1^\infty \frac{f(x)e^{iux}}{ix}\,dx$$

converges in square mean. According to Wiener [12], we define the transformation $Wf = g$ as

$$g(u) = Wf(u) = \frac{1}{2\pi}\left(\int_{-\infty}^{-1} + \int_1^\infty f(x)\frac{e^{iux}}{ix}\,dx + \int_{-1}^1 f(x)\frac{e^{iux}-1}{ix}\,dx\right)$$

and call g the Wiener transformation of f. We remark that the last term is used only to adjust the convergence of the integral.

Let S denote the space of rapid decreasing functions and let S' be its dual, the class of tempered distributions.

Proposition 3.1. *If $f \in B^2$, then f, $Wf \in S'$, and $(Wf)' = \check{f}$, where the derivative and $\check{f}$ is taken in the distributional sense.*

Proof. Let $h \in S$, then

$$
\langle (Wf)', h \rangle = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{-1} + \int_{1}^{\infty} f(x) \frac{e^{iux}}{ix} \, dx \right) h'(u) \, du
$$

$$
+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-1}^{1} f(x) e^{iux} \, dx \right) h(u) \, du
$$

$$
= \int_{-\infty}^{-1} + \int_{1}^{\infty} f(x) \check{h}(x) \, dx + \int_{-1}^{1} f(x) \check{h}(x) \, dx
$$

$$
= \langle f, \check{h} \rangle = \langle \check{f}, h \rangle .
$$

This implies that $(Wf)' = \check{f}$.

Theorem 3.2. *The Wiener transformation W is an isomorphism from B^2 onto V^2 with*

$$
\|W\| = \left(h(0) + \int_{1}^{\infty} \tilde{h}(x) \, dx \right)^{1/2} , \quad \|W^{-1}\| = (h(1))^{-1/2} ,
$$

where $h(x) = 2 \sin^2 x / \pi x^2$, $x \geq 0$, and $\tilde{h}(x) = \sup_{t \geq x} h(x)$.

Proof. Let $f \in B^2$, $g = W(f)$, it follows that

$$
\Delta_\varepsilon g(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) \frac{2 \sin \varepsilon x}{x} e^{iux} \, dx ,
$$

and

$$
\frac{1}{2\varepsilon} \int_{-\infty}^{\infty} |\Delta_\varepsilon g(u)|^2 \, du = \frac{1}{\varepsilon} \int_{-\infty}^{\infty} |f(x)|^2 \frac{\sin^2 \varepsilon x}{\pi x^2} \, dx
$$

$$
= \int_{-\infty}^{\infty} |f(x/\varepsilon)|^2 \frac{\sin^2 x}{\pi x^2} \, dx .
$$

If we let

$$
f_1(x) = \left(\tfrac{1}{2} \right) \left(|f(x)|^2 + |f(-x)|^2 \right) , \quad x > 0 ,
$$

and

$$
h(x) = \frac{2 \sin^2 x}{\pi x^2} , \quad x > 0 ,
$$

then Theorem 2.1, 2.3 imply that

$$
c_2 \sup_{T \geq 1} \frac{1}{2T} \int_{-T}^{T} |f(x)|^2 \, dx \leq \sup_{T \geq 1} \int_{-\infty}^{\infty} |f(Tx)|^2 \frac{2 \sin^2 x}{\pi x^2} \, dx
$$

$$
\leq c_1 \sup_{T \geq 1} \frac{1}{2T} \int_{-T}^{T} |f(x)|^2 \, dx ,
$$

where $c_1 = h(0) + \int_{1}^{\infty} \tilde{h}(x) \, dx$, $c_2 = h(1)$, i.e.

$$
c_1 \|f\|_{B^2}^2 \leq \|Wf\|_{V^2}^2 \leq c_1 \|f\|_{B^2}^2 .
$$

Hence W is an isomorphism, that $\|W\| = c_1^{1/2}$, $\|W^{-1}\| = c_2^{-1/2}$ follow from the fact that they are the best estimate for the inequalities.

It remains to show that W is a surjection. For this, we observe that $\Delta_\varepsilon^+ g = \tau_\varepsilon g - g \in L^2$ for all $\varepsilon \in R$. Define F_ε so that

$$(3.1) \qquad (e^{i\varepsilon x} - 1)F_\varepsilon(x) = \int_{-\infty}^{\infty} e^{-iux} \Delta_\varepsilon^+ g(u)\, du.$$

We claim that F_ε is independent of ε. Indeed, let $\eta \neq \varepsilon$, it is easy to check that

$$\Delta_\varepsilon^+ g(t + \eta) - \Delta_\varepsilon^+ g(t) = \Delta_\eta^+ g(t + \varepsilon) - \Delta_\eta^+ g(t).$$

By taking the Fourier transformation, we have

$$(e^{i\eta x} - 1)(e^{i\varepsilon x} - 1)F_\varepsilon(x) = (e^{i\eta x} - 1)(e^{i\varepsilon x} - 1)F_\eta(x).$$

This implies that F_ε is independent of ε. We denote it by F instead.

Let $f(x) = ixF(x)$, it is straightforward to check that $c_2\|f\|_{B^2}^2 \leq \|g\|_{V^2}^2 < \infty$, and hence $f \in B^2$. To show that $W(f) = g$, we observe that

$$\begin{aligned} \Delta_\varepsilon(Wf)(u) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) \frac{2\sin \varepsilon x}{ix} e^{iux}\, dx \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(x)(2\sin \varepsilon x)e^{iux}\, dx \\ &= \Delta_\varepsilon g(u), \end{aligned}$$

i.e. $\Delta_\varepsilon(Wf - g) = 0$. This implies that $Wf = g$ in V^2.

A function g on R has bounded variation if and only if $g \in V^1$ [7, 11]. In this case

$$\sup_{\varepsilon > 0} \frac{1}{\varepsilon} \int_{-\infty}^{\infty} |\Delta_+ \varepsilon g| = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_{-\infty}^{\infty} |\Delta_\varepsilon g| = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_{-\infty}^{\infty} |\Delta_\varepsilon g|$$

and equals the total variation of g. The Stieltjes integral $\int f\, dg$ for $g \in V^1$ is well known. The development of the correspondent theory for $g \in V^p$ is only partially successful (see Young [13]). Here we observe that for $f \in L^2$, $g \in L^2$ such that $g' \in L^2$,

$$\begin{aligned} (3.2) \qquad & \int_0^{\infty} \frac{1}{\varepsilon^2} \left(\int_{-\infty}^{\infty} \Delta_\varepsilon^+ f(u) \Delta_\varepsilon^- g(u)\, du \right) d\varepsilon \\ &= -\frac{1}{2\pi} \int_0^{\infty} \frac{1}{\varepsilon^2} \left(\int_{-\infty}^{\infty} (e^{i\varepsilon x} - 1)^2 \hat{f}(x)\hat{g}(-x)\, dx \right) d\varepsilon \\ &= -\frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(x)\hat{g}(-x) \left(\int_0^{\infty} \frac{(e^{i\varepsilon x} - 1)^2}{\varepsilon^2}\, d\varepsilon \right) dx \\ &= -\frac{2\ln 2}{2\pi} \int_{-\infty}^{\infty} (-ix)\hat{f}(x)\hat{g}(-x)\, dx \\ &= -2\ln 2 \int_{-\infty}^{\infty} f(x)\, dg(x). \end{aligned}$$

This leads us to define for $g \in V^2$,

$$(3.3) \qquad \int_{-\infty}^{\infty} f \, dg = c \int_0^{\infty} \frac{1}{\varepsilon^2} \left(\int_{-\infty}^{\infty} \Delta_\varepsilon^+ f(u) \Delta_\varepsilon^- g(u) \, du \right) d\varepsilon$$

whenever the integral converges, where $c = (-2 \ln 2)^{-1}$. In [12], Wiener has interpreted the integral $\int_{-\infty}^{\infty} e^{-iux} \, dg(u)$, $g \in V^2$ as first defining it as a formal integration by parts on finite intervals, and follows by taking the Cesero limit. The definition in (3.3) simplifies the meaning of the above integral. Moreover, we have

Proposition 3.3. *Let $g \in V^2$, then*

$$\int_{-\infty}^{\infty} e^{-iu(\bullet)} \, dg(u) = W^{-1}(g) \quad a.e.$$

Proof. Note that for almost all x,

$$\int_{-\infty}^{\infty} e^{-iux} \, dg(u) = c \int_0^{\infty} \frac{1}{\varepsilon^2} (e^{-i\varepsilon x} - 1) \left(\int_{-\infty}^{\infty} e^{-iux} \Delta_\varepsilon^- g(u) \, du \right) d\varepsilon$$

$$= -c \int_0^{\infty} \frac{1}{\varepsilon^2} (e^{-i\varepsilon x} - 1)^2 F(x) \, d\varepsilon$$

$$= ix F(x) = W^{-1}(g)(x)$$

(where $F(x)$ is defined as in the proof of Theorem 3.2).

In [3, Theorem IX], Beurling proved that the Fourier transformation of A^2 is the class of continuous functions g in

$$U^2 = \left\{ g : \|g\| = \|g\|_{L^2} + \int_0^{\infty} \frac{1}{\varepsilon^{3/2}} \|\Delta_\varepsilon g(u)\|_{L^2} \, d\varepsilon < \infty \right\}.$$

Moreover

$$\|f\|_{A^2} \le \frac{1}{\sqrt{2\pi}} \|\hat{f}\|_{U^2} \le 6\|f\|_{A^2}.$$

It is not difficult to see from Beurling's proof that the assumption that g is continuous is redundant. Indeed, let $f = \check{g}$, then

$$(e^{-i\varepsilon x} - 1) f(x) = \frac{1}{2\pi} \left(\int_{-\infty}^{\infty} e^{iux} \Delta_\varepsilon^+ g(u) \, du \right),$$

By [3, p. 14 and p. 24], $\|f\|_{A^2}^2 \le (1/2\pi)\|g\|_{U^2}^2$. This implies that $f \in A^2$ and hence in L^1. Therefore $g = \hat{f}$ is continuous.

If we define the duality of V^2 and U^2 by

$$\langle g, l \rangle = \int_{-\infty}^{\infty} \bar{l} \, dg, \qquad g \in V^2, \quad l \in U^2,$$

as in (3.3), then we have the following proposition.

Proposition 3.4. *Let* $g \in V^2$, $l \in U^2$, *then*

$$\langle g, l \rangle = \langle W^{-1}g, \check{l} \rangle.$$

Moreover $(U^2)^*$ *is isomorphic to* V^2 *under the above duality.*

Proof. The equality follows from a direct verification as in (3.2). The isomorphism of $(U^2)^*$ and V^2 is a consequence of Theorem 3.2, Beurling's Theorem and the above duality.

Remark 3.1. The duality and Wiener transformation can be summarized in the following diagram:

$$(A^2)^* \approx B^2 \underset{W}{\overset{W^{-1}}{\rightleftarrows}} V^2 \approx (U^2)^*$$

$$A^2 \underset{\vee}{\overset{\wedge}{\rightleftarrows}} U^2.$$

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